# **fc**-multicategories

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**Abstract**

What **fc**-multicategories are, and two uses for them.

## Introduction

**fc**-multicategories are a very general kind of two-dimensional structure, encompassing bicategories, monoidal categories, double categories and ordinary multicategories. Here we define what they are and explain how they provide a natural setting for two familiar categorical ideas. The first is the *bimodules* construction, traditionally carried out on suitably cocomplete bicategories but perhaps more naturally carried out on **fc**-multicategories. The second is *enrichment*: there is a theory of categories enriched in an **fc**-multicategory, which includes the usual case of enrichment in a monoidal category, the obvious extension of this to ordinary multicategories, and the less well known case of enrichment in a bicategory.

To finish we briefly indicate the wider context, including how the work below is just the simplest case of a much larger phenomenon and the reason for the name '**fc**-multicategory'.

## 1 What is an fc-multicategory?

An **fc**-multicategory consists of

- A collection of **objects** $x$, $x'$, ...

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- For each pair $(x, x')$ of objects, a collection of **vertical 1-cells** $\begin{array}{c} x \\ \downarrow \\ x' \end{array}$, denoted $f$, $f'$, ...
- For each pair $(x, x')$ of objects, a collection of **horizontal 1-cells** $x \longrightarrow x'$, denoted $m$, $m'$, ...
- For each $n \geq 0$, objects $x_0, \ldots, x_n, x, x'$, vertical 1-cells $f, f'$, and horizontal 1-cells $m_1, \ldots, m_n, m$, a collection of **2-cells**

$$\begin{tikzcd}
x_0 \arrow[r, "m_1"] \arrow[d, "f"'] & x_1 \arrow[r, "m_2"] & {} & \cdots & {} \arrow[r, "m_n"] & x_n \arrow[d, "f'"] \\
x \arrow[rrrrr, "m"'] & & & \Downarrow \arrow[u, phantom] & & x',
\end{tikzcd} \qquad (*)$$

denoted $\theta$, $\theta'$, ...
- **Composition** and **identity** functions making the objects and vertical 1-cells into a category
- A **composition** function for 2-cells, as in the picture

$$\begin{tikzcd}
\bullet \arrow[r, "m_1^1"] \arrow[d, "f_0"'] & \cdots \arrow[r, "m_1^{r_1}"] \arrow[d, phantom, "\Downarrow \theta_1"] & \bullet \arrow[r, "m_2^1"] \arrow[d] & \cdots \arrow[r, "m_2^{r_2}"] \arrow[d, phantom, "\Downarrow \theta_2"] & \bullet \arrow[d] & \cdots & \bullet \arrow[r, "m_n^1"] \arrow[d] & \cdots \arrow[r, "m_n^{r_n}"] \arrow[d, phantom, "\Downarrow \theta_n"] & \bullet \arrow[d, "f_n"] \\
\bullet \arrow[rr, "m_1"'] \arrow[d, "f"'] & & \bullet \arrow[rr, "m_2"'] & & \bullet \arrow[d, phantom, "\Downarrow \theta"] & \cdots & \bullet \arrow[rr, "m_n"'] & & \bullet \arrow[d, "f'"] \\
\bullet \arrow[rrrrrrrr, "m"'] & & & & & & & & \bullet
\end{tikzcd}$$

$\longmapsto$

$$\begin{tikzcd}
\bullet \arrow[r, "m_1^1"] \arrow[d, "f \circ f_0"'] & {} & \cdots \arrow[d, phantom, "\Downarrow \theta \circ (\theta_1, \theta_2, \ldots, \theta_n)"] & {} \arrow[r, "m_n^{r_n}"] & \bullet \arrow[d, "f' \circ f_n"] \\
\bullet \arrow[rrrr, "m"'] & & & & \bullet
\end{tikzcd}$$

$(n \geq 0, r_i \geq 0$, with $\bullet$'s representing objects)
- An **identity** function

$$x \xrightarrow{m} x' \qquad \longmapsto \qquad \begin{tikzcd}
x \arrow[r, "m"] \arrow[d, "1_x"'] \arrow[dr, phantom, "\Downarrow 1_m"] & x' \arrow[d, "1_{x'}"] \\
x \arrow[r, "m"'] & x'
\end{tikzcd}$$

such that 2-cell composition and identities obey associativity and identity laws.

**Examples**

a. Any double category gives an **fc**-multicategory, in which a 2-cell as at (∗) is a 2-cell

$$\begin{tikzcd}
x_0 \arrow[r, "m_n \circ \cdots \circ m_1"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & x_n \arrow[d, "f'"] \\
x \arrow[r, "m"'] & x'
\end{tikzcd}$$

in the double category.

b. Any bicategory gives an **fc**-multicategory in which the only vertical 1-cells are identity maps, and a 2-cell as at (∗) is a 2-cell

$$\begin{tikzcd}
x_0 \arrow[r, bend left, "m_n \circ \cdots \circ m_1"] \arrow[r, bend right, "m"'] \arrow[r, phantom, "\Downarrow"] & x_n
\end{tikzcd}$$

in the bicategory (with $x_0 = x$ and $x_n = x'$).

c. Any monoidal category gives an **fc**-multicategory in which there is one object and one vertical 1-cell, and a 2-cell

$$\begin{tikzcd}
\bullet \arrow[r, dash, "M_1"] \arrow[d, dash, "1"'] & \bullet \arrow[r, dash, "M_2"] & \bullet & \cdots & \bullet \arrow[r, dash, "M_n"] & \bullet \arrow[d, dash, "1"] \\
\bullet \arrow[rrrrr, dash, "M"'] \arrow[rrrrru, phantom, "\Downarrow"] & & & & & \bullet
\end{tikzcd} \qquad (\dagger)$$

is a morphism $M_n \otimes \cdots \otimes M_1 \longrightarrow M$.

d. Similarly, any ordinary multicategory gives an **fc**-multicategory: there is one object, one vertical 1-cell, and a 2-cell (†) is a map $M_1, \ldots, M_n \longrightarrow M$.

e. We define an **fc**-multicategory **Span**. Objects are sets, vertical 1-cells are functions, a horizontal 1-cell $X \longrightarrow Y$ is a diagram $\begin{tikzcd}[column sep=small] & M \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$, and a 2-cell inside

$$\begin{tikzcd}[column sep=small]
& M_1 \arrow[dl] \arrow[dr] & & M_2 \arrow[dl] \arrow[dr] & & \cdots & & M_n \arrow[dl] \arrow[dr] & \\
X_0 \arrow[dd, "f"'] & & X_1 & & {} & & {} & & X_n \arrow[dd, "f'"] \\
& & & & M \arrow[dllll] \arrow[drrrr] & & & & \\
X & & & & & & & & X'
\end{tikzcd} \qquad (\ddagger)$$

is a function $\theta$ making

$$\begin{tikzcd} & M_n\circ\cdots\circ M_1 \arrow[dl] \arrow[dd, "\theta"] \arrow[dr] & \\ X_0 \arrow[dd, "f"'] & & X_n \arrow[dd, "f'"] \\ & M \arrow[dl] \arrow[dr] & \\ X & & X' \end{tikzcd}$$

commute, where $M_n\circ\cdots\circ M_1$ is the limit of the top row of (‡). Composition is defined in the obvious way.

# 2 Bimodules

Bimodules have traditionally been discussed in the context of bicategories. Thus given a bicategory $\mathcal{B}$, we construct a new bicategory $\mathbf{Bim}(\mathcal{B})$ whose 1-cells are bimodules in $\mathcal{B}$ (see [CKW] or [Kos]). The drawback is that to do this, we must make certain assumptions about the behaviour of local coequalizers in $\mathcal{B}$.

However, the **Bim** construction extends to **fc**-multicategories, and working in this context allows us to drop all the technical assumptions: we therefore obtain a functor $\mathbf{Bim} : \mathbf{fc\text{-}Multicat} \longrightarrow \mathbf{fc\text{-}Multicat}$. The definition is rather dry, so we omit it here and just give a few examples; the reader is referred to [Lei2, 2.6] for further details.

**Examples**

a. Let $V$ be the **fc**-multicategory coming from the monoidal category $(\mathbf{Ab}, \otimes)$ (see (c) above). Then $\mathbf{Bim}(V)$ has

**objects:** rings

**vertical 1-cells:** ring homomorphisms

**horizontal 1-cells** $R \longrightarrow S$**:** $(S, R)$-bimodules

**2-cells:** A 2-cell

$$\begin{tikzcd} R_0 \arrow[r, "M_1"] \arrow[d, "f"'] & R_1 \arrow[r, "M_2"] & {} \arrow[r, phantom, "\cdots"] \arrow[d, phantom, "\Downarrow\theta"] & {} \arrow[r, "M_n"] & R_n \arrow[d, "f'"] \\ R \arrow[rrrr, "M"'] & & & & R' \end{tikzcd}$$

is a multi-additive map $M_n \times \cdots \times M_1 \xrightarrow{\theta} M$ of abelian groups such that

$$\begin{aligned} \theta(r_n.m_n, m_{n-1}, \ldots) &= f(r_n).\theta(m_n, m_{n-1}, \ldots) \\ \theta(m_n.r_{n-1}, m_{n-1}, \ldots) &= \theta(m_n, r_{n-1}.m_{n-1}, \ldots) \end{aligned}$$

etc.

b. If $V$ is the **fc**-multicategory **Span** then **Bim**$(V)$ has

**objects:** monads in **Span**, i.e. small categories

**vertical 1-cells:** functors

**horizontal 1-cells** $\mathbb{C} \longrightarrow \mathbb{C}'$**:** profunctors (i.e. functors $\mathbb{C}^{\mathrm{op}} \times \mathbb{C}' \longrightarrow$ **Set**)

**2-cells:** A 2-cell

$$
\begin{tikzcd}
\mathbb{C}_0 \arrow[r, "M_1"] \arrow[d, "F"'] & \mathbb{C}_1 \arrow[r, "M_2"] & {} \arrow[r, phantom, "\cdots"] \arrow[d, phantom, "\Downarrow"] & {} \arrow[r, "M_n"] & \mathbb{C}_n \arrow[d, "F'"] \\
\mathbb{C} \arrow[rrrr, "M"'] & & & & \mathbb{C}'
\end{tikzcd}
$$

is a natural family of functions

$$M_1(c_0, c_1) \times \cdots \times M_n(c_{n-1}, c_n) \longrightarrow M(Fc_0, F'c_n),$$

one for each $c_0 \in \mathbb{C}_0, \ldots, c_n \in \mathbb{C}_n$.

c. Let $V$ be the **fc**-multicategory coming from a bicategory $\mathcal{B}$ with nicely-behaved local coequalizers. If we discard the non-identity vertical 1-cells from **Bim**$(V)$ then we obtain the **fc**-multicategory coming from the traditional bicategory **Bim**$(\mathcal{B})$—e.g. in (a), we get the bicategory of rings and bimodules.

# 3 Enrichment

We define what a 'category enriched in $V$' is, for any **fc**-multicategory $V$. This generalizes the established definitions for monoidal categories and bicategories.

Fix an **fc**-multicategory $V$. A **category** $C$ **enriched in** $V$ consists of

- a set $C_0$ ('of objects')
- for each $a \in C_0$, an object $C[a]$ of $V$
- for each $a, b \in C_0$, a horizontal 1-cell $C[a] \xrightarrow{C[a,b]} C[b]$ in $V$
- for each $a, b, c \in C_0$, a 'composition' 2-cell

$$
\begin{tikzcd}
C[a] \arrow[r, "{C[a,b]}"] \arrow[d, "1"'] & C[b] \arrow[r, "{C[b,c]}"] \arrow[d, phantom, "\Downarrow comp_{a,b,c}"] & C[c] \arrow[d, "1"] \\
C[a] \arrow[rr, "{C[a,c]}"'] & & C[c]
\end{tikzcd}
$$

- for each $a \in C_0$, an 'identity' 2-cell

$$\begin{tikzcd} C[a] \arrow[r, equal] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow id_a"] & C[a] \arrow[d, "1"] \\ C[a] \arrow[r, "{C[a,a]}"'] & C[a] \end{tikzcd}$$

(where the equality sign along the top denotes a string of 0 horizontal 1-cells)

such that *comp* and *id* satisfy associativity and identity axioms.

*Remark:* We haven't used the vertical 1-cells of $V$ in any significant way, but we would do if we went on to talk about *functors* between enriched categories (which we won't here).

**Examples**

a. Let $V$ be (the **fc**-multicategory coming from) a monoidal category. Then the choice of $C[a]$'s is uniquely determined, so we just have to specify the set $C_0$, the $C[a,b]$'s, and the maps $C[b,c] \otimes C[a,b] \longrightarrow C[a,c]$ and $I \longrightarrow C[a,a]$. This gives the usual notion of enriched category.

b. If $V$ is an ordinary multicategory then we obtain an obvious generalization of the notion for monoidal categories: so a category enriched in $V$ consists of a set $C_0$, an object $C[a,b]$ of $V$ for each $a, b$, and suitable maps $C[a,b], C[b,c] \longrightarrow C[a,c]$ and $\cdot \longrightarrow C[a,a]$ (where $\cdot$ denotes the empty sequence).

c. If $V$ is a bicategory then we get the notion of Walters et al (see [BCSW], [CKW], [Wal]).

d. Let $D$ be a category enriched in $(\mathbf{Ab}, \otimes)$. Then we get a category $C$ enriched in $\mathbf{Bim}(\mathbf{Ab})$:

   - $C_0 = D_0$ (= objects of $D$)
   - $C[a]$ is the ring $D[a,a]$ (whose multiplication is composition in $D$)
   - $C[a,b]$ is the abelian group $D[a,b]$ acted on by $C[a] = D[a,a]$ (on the right) and $C[b] = D[b,b]$ (on the left)
   - composition and identities are as in $D$.

   So the passage from $D$ to $C$ is basically down to the fact that composition makes $D[a,a]$ into a ring and $D[a,b]$ into a $(D[b,b], D[a,a])$-bimodule. It's a very mechanical process, and in fact for general $V$ there's a functor

$$(\text{categories enriched in } V) \longrightarrow (\text{categories enriched in } \mathbf{Bim}(V)).$$

e. An example of a category enriched in **Bim**(**Span**) (= categories + functors + profunctors ... ): $C_0$ is $\mathbb{N}$, $C[n]$ is the category of $n$-dimensional real differentiable manifolds and diffeomorphisms, and the profunctor $C[m,n]$ is the functor

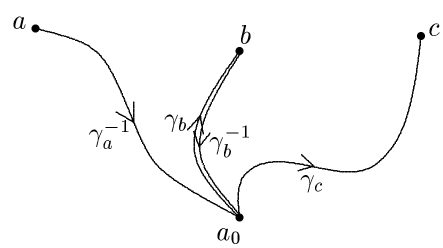
$$\begin{array}{rcl} C[m]^{\mathrm{op}} \times C[n] & \longrightarrow & \mathbf{Set} \\ (M,N) & \longmapsto & \{\text{differentiable maps } M \longrightarrow N\}. \end{array}$$

f. Let **ParBjn** be the sub-**fc**-multicategory of **Span** in which all horizontal 1-cells are of the form $(X \longleftarrow\!\!< M >\!\!\longrightarrow Y)$: so this 1-cell is a partial bijection between $X$ and $Y$. Let $S$ be a set and $(C_i)_{i \in I}$ a family of subsets. Then we get a category $C$ enriched in **ParBjn**:

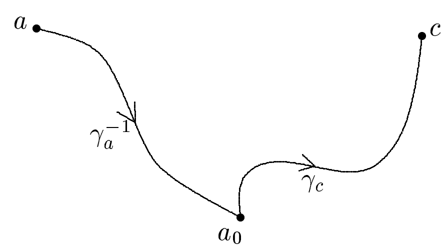
- $C_0 = I$
- $C[i] = C_i$
- $C[i,j] = (C_i \longleftarrow\!\!< C_i \cap C_j >\!\!\longrightarrow C_j)$
- $comp_{i,j,k}$ is the inclusion $C_i \cap C_j \cap C_k \subseteq C_i \cap C_k$
- $id_i$ is the inclusion $C_i \subseteq C_i \cap C_i$.

g. Fix a topological space $A$. Suppose $A$ is nonempty and path-connected; choose a basepoint $a_0$ and a path $\gamma_a : a_0 \longrightarrow a$ for each $a \in A$. Then we get a category $C$ enriched in the homotopy bicategory $V$ of $A$ (where $V$ consists of points of $A$, paths in $A$, and homotopy classes of path homotopies in $A$):

- $C_0 = A$
- $C[a] = a$
- $C[a,b]$ is $a \xrightarrow{\gamma_a^{-1}} a_0 \xrightarrow{\gamma_b} b$
- composition $C[b,c] \circ C[a,b] \longrightarrow C[a,c]$ is the (homotopy class of the) obvious homotopy from




to




- identities work similarly.

## The wider context

Given a monad $T$ on a category $\mathcal{E}$ (with certain properties), there's a category of **$T$-multicategories** (see [Lei1], [Bur] or [Her]). For example:

| $(\mathcal{E}, T)$ | $T$-multicategories |
|---|---|
| $(\mathbf{Set}, id)$ | categories |
| $(\mathbf{Set}, \text{free monoid})$ | ordinary multicategories |
| $(\mathbf{Graph}, \underline{\text{f}}\text{ree } \underline{\text{c}}\text{ategory})$ | **fc**-multicategories |

where $\mathbf{Graph} = [(\bullet \rightrightarrows \bullet), \mathbf{Set}]$.

Moreover, if one defines a **$T$-graph** to be a diagram

$$\begin{tikzcd} & C_1 \arrow[dl] \arrow[dr] & \\ T(C_0) & & C_0 \end{tikzcd}$$

in $\mathcal{E}$, then there's a forgetful functor $T$-**Multicat** $\longrightarrow$ $T$-**Graph**, this has a left adjoint, and the adjunction is monadic. Write $\mathcal{E}' = T\text{-}\mathbf{Graph}$ and $T'$ for the induced monad on $\mathcal{E}'$. Then we can also discuss $T'$-multicategories, and in fact there's a theory of $T$-multicategories enriched in a $T'$-multicategory.

The simplest case is $(\mathcal{E}, T) = (\mathbf{Set}, id)$: then $(\mathcal{E}', T') = (\mathbf{Graph}, \text{free category})$, so we have a theory of categories enriched in an **fc**-multicategory. This is just the theory we discussed above.

A full explanation of these ideas can be found in [Lei2].

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